ON THREE-DIMENSIONAL HYPERSONIC FLOW PAST SLENDER AIRFOILS

(O PROSTRANSTVENNOM GIPERZVUKOVOM TECHENII OKOLO TONKIKH KRYL'EV) (*)

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In flow of an inviscid hypersonic stream past slender airfoils of not too small-aspect ratio with sharp leading edges, the so-called "theory of slices" [1] is applicable, according to which the flow in each plane in the direction of the stream can be considered independently, as in flow past a two-dimensional profile.

In this paper we give a formulation of the problems of nonviscous flow past a slender airfoil with blunt leading edges and of the flow of viscous gas past an airfoil with sharp leading edges.

It is shown that in the case when the thickness of the entropic or the boundary layer is comparable with or exceeds the thickness of the airfoil, the slice theory in its usual form is inapplicable. This theory can be used to calculate the flow outside the entropic or the boundary layer. For the case of flow of a viscous gas the slice theory can be applied to the whole flow for small values of the parameter $\varepsilon = (\varkappa - 1) / (\varkappa + 1)$, where $\varkappa$ is the adiabatic exponent.

An example is presented of a solution for the case of nonviscous flow past a delta airfoil with blunt leading edges under the assumption of isentropic flow inside the entropic layer. The self-similar solution is considered for the problem of flow of a viscous gas past a triangular plate with sharp leading edges under the strong interaction conditions between the inviscid stream and the boundary layer.

At the present time there exist studies of the influence of viscosity [1] and slight bluntness of the leading edges of bodies [2] for the cases of two-dimensional and axisymmetric flows. In papers [3 and 4] there is presented a study of the influence of bluntness, and in [5] a study of the influence of viscosity on three-dimensional hypersonic flow past slender prolate bodies under the condition that the thickness of the entropic [6] or of the boundary layer be comparable with or exceed the thickness of the body. In the present paper we consider under the same condition the influence of bluntness of the leading edges and the influence of viscosity on three-dimensional hypersonic flow past slender airfoils.

1. We make use of a Cartesian system of coordinates (Fig.1) Lx, Ly, Lz, where L is a characteristic length, the x-axis is directed along the velocity

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vector U_∞ of the undisturbed stream. We define the following notation: uU_∞, vU_∞, wU_∞ are the components of velocity along the x, y and z axes, respectively, $p\rho_\infty U_\infty^2$ is the pressure, $\rho\rho_\infty$ is the density (ρ_∞ is the density of the undisturbed stream), Ld, $L\tau$, $L\delta$ are quantities characterizing the respective dimensions of bluntness, thickness of the airfoil, and thickness of the entropic or the boundary layer (it is assumed that

$$d \ll 1, \quad \tau \ll 1, \quad \delta \ll 1, \quad \delta \gtrsim \tau, \quad M_\infty\tau \gtrsim 1,$$

where M_∞ is the Mach number of the undisturbed stream).

Let the equation of the surface of the airfoil have the form $y = \tau f(x,z)$, whilst the equation of the outer boundary of the entropic or the boundary layer is $y = \tau\varphi(x,z)$; we shall assume here that $f \sim \varphi \sim 1$.

Outside the entropic or the boundary layer from the estimates [1]

$$u = 1 + o(\tau^2), \quad v \sim \tau, \quad w \sim \tau^2, \quad p \sim \tau^2, \quad \rho \sim 1$$

it follows that in each plane $z = z_0 =$ const the equations of motion with relative error of order τ^2 reduce to the equations of one-dimensional unsteady flow of gas displaced by a piston

$$\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \quad \frac{\partial \rho}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \quad \frac{\partial}{\partial x}\frac{p}{\rho^\varkappa} + v\frac{\partial}{\partial y}\frac{p}{\rho^\varkappa} = 0 \quad (1.1)$$

The boundary conditions on the boundary of the entropic or the boundary layer and on the surface of the shock wave $y = \tau\psi(x,z)$ may be written down (the subscript x denotes partial differentiation with respect to x)

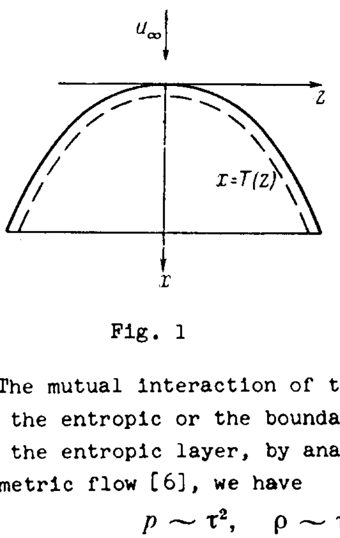

Fig. 1

$$v = \frac{2\tau\psi_x}{\varkappa+1}\left(1 - \frac{1}{M_\infty^2\tau^2\psi_x^2}\right)$$

$$p = \frac{2\tau^2\psi_x^2}{\varkappa+1}\left(1 - \frac{\varkappa - 1}{2\varkappa M_\infty^2\tau^2\psi_x^2}\right) \quad (1.2)$$

$$\rho = \frac{\varkappa+1}{\varkappa-1}\left(1 + \frac{2}{(\varkappa-1)M_\infty^2\tau^2\psi_x^2}\right)^{-1}$$

for $y = \tau\psi(x, z_0)$

$$v = \tau\varphi_x \quad \text{for } y = \tau\varphi(x, z_0)$$

The mutual interaction of the cross-sections $z =$ const is accomplished via the entropic or the boundary layer, the flow in which is three-dimensional. For the entropic layer, by analogy with the cases of two-dimensional or axisymmetric flow [6], we have

$$p \sim \tau^2, \quad \rho \sim \tau^{2/\varkappa}, \varepsilon^{-1}, \quad u \sim 1, \quad v \sim \tau, \quad w \sim \tau^{2-2/\varkappa}$$

The estimate for w is obtained from consideration of the flow close to the leading edge of the airfoil, making use of the principle of local sweep. Similar estimates hold for the boundary layer (it is assumed that the temperature in the boundary layer has the order of the stagnation temperature of the undisturbed stream [1])

$$p \sim \tau^2, \quad \rho \sim \tau^2/\varepsilon, \quad u \sim 1, \quad v \sim \tau, \quad w \sim \varepsilon$$

The estimate for w is obtained from the projection of the momentum equation on the z-axis. There is an essential difference in the behavior of w as ϵ tends to zero $(\varkappa \to 1)$ in the case of the entropic layer (w increases) and in the case of the boundary layer $(w \to 0)$. From the projection of the momentum equation on the y-axis, we have, for the differential pressure Δp across the entropic or the boundary layer, respectively,

$$p^{-1}\Delta p \sim \tau^{2/\varkappa}\varepsilon^{-1}, \qquad p^{-1}\Delta p \sim \tau^2\varepsilon^{-1}$$

To this order of error the pressure inside these layers can be assumed to depend only on x and z.

2. Let us consider the nonviscous flow in the three-dimensional entropic layer. The equations of motion are written down in the form

$$\begin{gathered}
\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial x} = 0 \\
\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} = 0 \\
p = p(x, z), \ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \\
u\frac{\partial}{\partial x}\frac{p}{\rho^\varkappa} + v\frac{\partial}{\partial y}\frac{p}{\rho^\varkappa} + w\frac{\partial}{\partial z}\frac{p}{\rho^\varkappa} = 0
\end{gathered} \tag{2.1}$$

The boundary conditions, according to which the surface of the body and the outer surface of the entropic layer are stream surfaces, have the form

$$\begin{gathered}
v = \tau\,(uf_x + wf_z) \quad \text{for } y = \tau f\,(x, y) \\
v = \tau\,(u\varphi_x + w\varphi_z) \quad \text{for } y = \tau\varphi\,(x, z)
\end{gathered} \tag{2.2}$$

From the requirement for the thickness δ of the entropic layer $(\delta \gtrsim \tau)$ we find, using the equations of continuity and of isentropic condition, a relation between τ and the characteristic bluntness dimension d

$$\tau \lesssim (\varepsilon d)^{\varkappa/(2+\varkappa)} \tag{2.3}$$

Equations (2.1) and (2.2) are solved together with equations (1.1) and (1.2), taking account of the continuity of pressure on the surface $y = \tau\varphi(x,z)$, which is determined from the solution.

The computation starts from a certain surface

$$x = T(z), \ \tau f(T(z), z) \leqslant y \leqslant \tau\psi\ (T(z), z)$$

The curve $x = T(z)$, by assumption, is separated from the leading edge by a distance of the order of the characteristic dimension of bluntness d. It is assumed that the flow parameters are calculated on this surface. The calculation can be achieved, foe example, by making use of the principle of local sweep, as for the leading portion of a two-dimensional blunted profile.

Let us consider further the streamline flow past a delta airfoil, assuming the flow in the three-dimensional entropic layer is isentropic, which is valid if all the gas in the entropic layer passed through a shock wave of uniform intensity. This is approximately fulfilled if the bluntness occurs in the form of a wedge with a large included angle or a plane face.

For the stipulated type of bluntness the assumption of the flow being isentropic in the entropic layer is equivalent to the hypothesis that the outer "transitional" part of the region of the entropic layer, in which the vorticity is essential and the flow in which cannot be taken to be isentropic, is far thinner than the whole entropic layer and therefore can be approximately taken as a surface of tangential discontinuity, coinciding with the outer boundary of the entropic layer $y = \tau\varphi(x,z)$. The position and intensity of the tangential discontinuity, which separates the entropic layer from the rest of the flow, are determined from the solution.

It should be noted that the specified hypothesis becomes more exact as $\varkappa \to 1$, when the shock wave comes closer to the body. Then it is clear that relation (2.3) must remain valid.

To the specified accuracy the system of equations (2.1) can be written as

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad p = p\,(x,\,z) \tag{2.4}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \quad \frac{u^2 + w^2}{2} + \frac{\varkappa}{\varkappa - 1}\,\frac{p}{\rho} = \frac{1}{2} + \frac{1}{(\varkappa - 1)\,M_\infty^2}, \quad \frac{p}{\rho^\varkappa} = \text{const}$$

Here the first three equations express the absence of vorticity, which follows from the assumption of the flow being isentropic. From (2.4) it follows that

$$u = u\,(x,\,z), \quad w = w\,(x,\,z), \quad p = p\,(x,\,z), \quad \rho = \rho\,(x,\,z) \tag{2.5}$$

In the equation of continuity the first and third terms depend on x and z ; hence the second term is a function of those same variables; whence it follows that the dependence of v on y is linear. From Equation (2.2) we have

$$\frac{\partial v}{\partial y} = \frac{u\,(\varphi_x - f_x) + w\,(\varphi_z - f_z)}{\varphi - f} \tag{2.6}$$

Substituting (2.6) in the equation of continuity and making use of the equation of isentropic condition and Bernoulli's integral, we reduce system (2.4) to the form

$$\frac{\partial}{\partial x}\,[\rho u\,(\varphi - f)] + \frac{\partial}{\partial z}\,[\rho w\,(\varphi - f)] = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{2.7}$$

$$\rho = \rho_0 \left(1 - \frac{u^2 + w^2}{U_m^2}\right)^{1/(\varkappa - 1)}, \qquad U_m^2 = U_\infty^2 \left(1 + \frac{2}{(\varkappa - 1)\,M_\infty^2}\right)$$

Here ρ_0 is the stagnation density in the entropic layer; it is constant by assumption. In order to complete system (2.7), it is necessary to specify the functional dependence between the pressure p (and also between the density, since $p\,/\,\rho^\varkappa = \text{const}$) and the equation of the outer boundary of the entropic layer $y = \varphi(x,z)$. This dependence is determined from the solution of the system (1.1) with the boundary conditions (1.2) .

Equations (2.7) satisfy the boundary conditions on the curve $x = T(z)$, where all the parameters of the flow are specified.

We notice that the assumption of the flow being isentropic offeres the possibility of "averaging" of Equations (2.1) with respect to the y coordinate in the simplest way. Other variants of averaging are also possible with allowance for variability of entropy, which would lead to more complicated equations than (2.7).

The system (2.7) calls to mind the system of equations of plane flow of a compressible gas, which facilitates analysis. The solution of this system (with a known functional relationship between ρ and φ) can be accomplished by the method of characteristics.

Let us construct a particularly simple example of the solution, assuming the thickness of the entropic layer to be everywhere a constant quantity $\varphi - f = C$, where C is a constant, determined from computation of the flow close to the leading edges. In this case the system (2.7) with the boundary conditions which, making use of the symmetry of the flow, are set on the curve $x = T(z)$ for $z > 0$ and on the line $z = 0$, takes the form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \rho = \rho_0\left(1 - \frac{u^2 + w^2}{U_m^2}\right)^{1/(\varkappa-1)} \qquad (2.8)$$

$$u = u_0(z), \quad w = w_0(z) \quad \text{for } x = z \operatorname{tg} \chi_\Delta, \qquad w = 0 \quad \text{for } z = 0$$

In the equation of the curve $x = T(z)$ with accuracy of order d it is assumed that this curve coincides with the leading edge, the angle of sweep of which is denoted by χ_Δ. From the solution of the system (2.8) we determine $\rho(x,z)$ and $p(x,z)$; and from the pressure so calculated, by using the functional relationship between p and φ we construct the outer boundary of the entropic layer $y = \varphi(x,z)$, whilst from the equation $f = \varphi - C$ we determine the surface of the body. By this method, then, the inverse problem is solved.

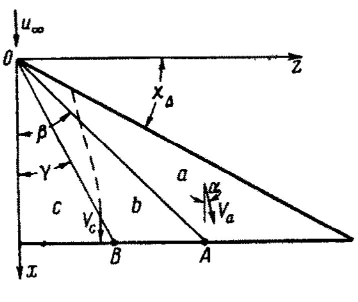

Fig. 2

For flow past a delta airfoil bluntness of shape of which is constant along the span, the flow parameters on the curve $x = T(z)$ calculated by using the principle of swept wings, are constant (with the exception of a small region in the neighborhood of the vertex of the airfoil). In (2.8) we can set u_0 = const and w_0 = const . As is shown by calculations on the curve $x = T(z)$ the velocity component perpendicular to this curve exceeds the velocity of sound. Besides this, it is not difficult to show that the velocity vector on this curve makes a positive angle with the x-axis when $z > 0$, i.e. $w > 0$. Bearing in mind these circumstances, it is not difficult to construct the flow pattern on the basis of the solution of Cauchy's problem for Equation (2.8). The flow pattern (for half of the airfoil) is as shown in Fig.2. A region a of uniform flow adjoining the leading edge corresponds to the case of flow past a swept leading edge plate of infinite span. Next there follows a region b of Prandtl-Meyer flow and then again comes a region c of uniform flow. As follows from construction of the solution, the angle between the velocity vector $\mathbf{V}_a$ in the region a and the ray OA is equal to the Mach angle in region a, and similarly the angle between $\mathbf{V}_c$ and the ray OB is equal to the Mach angle in region c . This simple example illustrates the appearance on the delta airfoil of regions of rarefaction with pressure less than the pressure which is given by calculation according to the theory of swept leading edge plate of infinite span. The results obtained, as may be expected, do not depend upon the simplifying assumptions made.

The appearance of regions of rarefaction follows from the fact that close to the leading edges the velocity component w is directed from the middle towards the edges of the airfoil, in accordance with which the streamlines (broken line in Fig.2) are concave towards the x-axis.

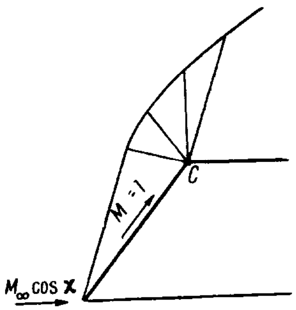

Fig. 3

Let us present values of the flow parameters, calculated according to the solution described for $\varkappa = 1.4$, $M_\infty = \infty$ and three values of the angle of sweep χ_Δ

χ_Δ	M_a	M_c	α	β	γ
30°	3.45	4.0	6°40′	24°51′	14°29′
45°	4.57	5.31	7°05′	19°43′	10°57′
60°	7.24	8.7	6°20′	13°41′	6°33′

The leading edge, by assumption, is a wedge (Fig.3) such that, in the flow past it, perpendicular to the leading edge, the Mach number behind the shock = 1 (the semi-angle of the edge is equal to 45.5°). It is assumed that the curve $x = T(z)$ passes close to the shoulder C of the wedge (Fig.3) and therefore the velocity component normal to the leading edge on the curve $x = T(z)$ is taken from the Prandtl-Meyer solution immediately next to the point C after the expansion. The velocity component along the tangent to the curve $x = T(z)$ is equal to its value in the undisturbed flow. The values given above for M_a and M_c are the Mach numbers in the corresponding regions, whilst the remaining parameters are defined in Fig.2.

From the given data it follows that the deviations of the streamlines from the straight lines z = const are not large. These same data enable us also to estimate the relative dimensions of the regions a, b and c for different values of χ_Δ.

We notice moreover a simple solution of Equations (2.7) for the case of flow past a plane triangular plate at an angle of attack, when $f = x$ and the pressure is calculated according to Newton's formula $p = \varphi_x^2$. Moreover, if we neglect the thickness of the entropic layer near the leading edge in comparison with its thickness at the middle portion of the airfoil, then in Equations (2.7) we can take $\varphi = x\Phi(\zeta)$, $u = u(\zeta)$, $w = w(\zeta)$, $\rho = \rho(\zeta)$, where $\zeta = z/x$. Then Equations (2.7) reduce to a system of ordinary differential equations with respect to ζ .

3. It is essential to note that in the case when the thickness of the entropic layer is comparable with or exceeds the thickness of the body, the lift Y for slender elongated bodies is diminished in comparison with its value Y_0 for pointed bodies.

From [3], where streamline flow past slender blunt elongated bodies is considered under the assumption [2] concerning the strong compression of the gas behind the shock wave, it follows generally that $Y/Y_0 = 0$ From [4], where a refinement of the results of [3] is given with the help of introduction of the entropic layer [6], it follows that $Y / Y_0 \sim \tau^{2/\varkappa}/\varepsilon$. For the case under consideration of flow past planar bodies of airfoil type the bluntness increases the lift. Papers [7 and 8], where flow past a plane blunt plate at a small angle of attack is worked out by the method of perturbations, confirm the stated conclusion.

This difference arises from the fact that for deflection at a small angle in the case of an airfoil the entropic layer is displaced by the body, whilst in the case of a slender elongated body the boundary of the enropic layer is scarcely altered.

4. Let us consider the flow of a viscous hypersonic stream past slender airfoils with sharp leading edges. We introduce the following supplementary notation: $\mu\mu_0$ is the coefficient of viscosity (μ_0 is the coefficient of viscosity corresponding to the stagnation temperature of the unperturbed stream), σ is the Prandtl number, hU_∞^2 is the enthalpy.

The equations of the three-dimensional boundary layer have the form

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)\frac{1}{R_*}, \quad p = \frac{\varkappa - 1}{\varkappa}\rho h$$

$$\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right)\frac{1}{R_*}, \quad R_* = \frac{\rho_\infty U_\infty L}{\mu_0}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \qquad p = p(x, z), \qquad \mu = \mu(h) \tag{4.1}$$

$$\rho u \frac{\partial}{\partial x}\left(h + \frac{u^2 + w^2}{2}\right) + \rho v \frac{\partial}{\partial y}\left(h + \frac{u^2 + w^2}{2}\right) + \rho w \frac{\partial}{\partial z}\left(h + \frac{u^2 + w^2}{2}\right) =$$
$$= \frac{1}{R_*}\frac{\partial}{\partial y}\left(\frac{\mu}{\sigma}\frac{\partial h}{\partial y}\right) + \frac{1}{R_*}\frac{\partial}{\partial y}\left(\mu \frac{\partial}{\partial y}\frac{u^2 + w^2}{2}\right)$$

The flow outside the boundary layer satisfies the system of equations (1.1) and the boundary conditions (1.2). It is assumed that the thickness of the body does not exceed the thickness of the boundary layer $\delta \sim (\varepsilon/R_*)^{1/4}$. On the surface separating the boundary layer from the inviscid stream $y = \tau\varphi(x, z)$ we have to satisfy the conditions of continuity of pressure and of the vertical component of velocity. The temperature on the outer surface of the boundary layer is assumed to be equal to zero. On the body, the equations of the surface of which is $y = \tau f(x\ z)$, we have to satisfy the usual conditions of adhesion and the equality of temperature of the gas and the wall (or the condition for the heat flow into the wall).

As an example of three-dimensional viscous hypersonic flow let us consider the flow past of plane triangular plate with sharp leading edges, with a given surface temperature or heat insulation, at $M_\infty = \infty$. We shall seek the equations of the outer surface of the boundary layer, the surface of the shock wave and the remaining parameters in the region of inviscid flow in the form

$$y = \delta\varphi(x, z) = \delta x^{3/4}\Phi(\zeta), \quad y = \delta\psi(x, z) = \delta x^{3/4}\Psi(\zeta) \tag{4.2}$$

$$p = \frac{\delta^2 P_0(\zeta, \eta)}{\sqrt{x}}, \quad \rho = R_0(\zeta, \eta), \quad v = \frac{\delta V_0(\zeta, \eta)}{x^{1/4}}, \quad \delta = \frac{1}{R_*^{1/4}}$$

$$\zeta = \frac{z}{x}, \quad \eta = \frac{y}{\delta x^{3/4}}$$

Equations (4.1) of inviscid flow and the boundary conditions at the shock wave (1.2) and on the surface of the boundary layer take the form

$$R_0\left(V_0 - \frac{3\eta}{4}\right)\frac{\partial V_0}{\partial \eta} - R_0\zeta\frac{\partial V_0}{\partial \zeta} - \frac{R_0 V_0}{4} + \frac{\partial P_0}{\partial \zeta} = 0$$

$$\left(V_0 - \frac{3\eta}{4}\right)\frac{\partial R_0}{\partial \eta} + R_0\frac{\partial V_0}{\partial \eta} - \zeta\frac{\partial R_0}{\partial \zeta} = 0$$

$$\left(V_0 - \frac{3\eta}{4}\right)\frac{\partial}{\partial \eta}\frac{P_0}{R_0^{\varkappa}} - \zeta\frac{\partial}{\partial \zeta}\frac{P_0}{R_0^{\varkappa}} - \frac{P_0}{2R_0^{\varkappa}} = 0 \tag{4.3}$$

$$V_0 = \frac{2}{\varkappa + 1}\left(\frac{3}{4}\Psi - \zeta\frac{d\Psi}{d\zeta}\right), \quad P_0 = \frac{2}{\varkappa + 1}\left(\frac{3}{4}\Psi - \zeta\frac{d\Psi}{d\zeta}\right)^2$$

$$R_0 = \frac{\varkappa+1}{\varkappa-1} \quad \text{for} \quad \eta = \Psi(\zeta) \tag{4.3 cont.}$$

$$V_0 = \frac{3}{4}\Phi - \zeta\frac{d\Phi}{d\zeta} \quad \text{for} \quad \eta = \Phi(\zeta)$$

For the boundary layer we seek the solution in the form

$$u = U(\eta, \xi), \quad w = W(\eta, \zeta), \quad v = x^{-1/4}\delta V(\eta, \zeta), \quad h = h(\eta, \zeta) \tag{4.4}$$

$$\rho = x^{-1/2}\delta^2 R(\eta, \zeta), \quad p = x^{-1/2}\delta^2 \; P(\zeta), \quad P(\zeta) = P_0(\zeta, \Phi(\zeta)), \quad \delta = R_*^{-1/4}$$

The boundary conditions for the system of equations of the boundary layer take the form

$$V = 0,\ U = 0,\ W = 0,\ h = h_b = \text{const} \ (\text{or} \quad \partial h / \partial\eta = 0) \quad \text{for} \quad \eta = 0$$

$$V = V_0 = \frac{3}{4}\Phi_0 - \zeta\frac{d\Phi_0}{d\zeta} \tag{4.5}$$

$$U = 1, \quad W = 0, \quad h = 0 \quad \text{for} \quad \eta = \Phi(\zeta), \ -\zeta_0 \leqslant \zeta \leqslant \zeta_0 = \cot\chi_\Delta$$

Here χ_Δ —is, as before, the angle of sweep of the leading edges.

Substituting Expressions (4.4) in Equations (4.1), we obtain a system of equations in two independent variables η and ζ, which have to be solved together with Equations (4.3) while fulfilling the boundary conditions (4.5).

For the sake of simplicity we use for the pressure the formula of the tangent wedges, which introduces in the plane case an insignificant error of the order of a few per cent [1]

$$p = \frac{\varkappa+1}{2}\delta^2\left(\frac{\partial\varphi}{\partial x}\right)^2, \qquad P(\zeta) = \frac{\varkappa+1}{2}\left(\frac{3\Phi}{4} - \frac{\zeta d\Phi}{d\zeta}\right)^2 \tag{4.6}$$

Use of (4.6) makes it possible to avoid solving the system (4.3). We shall assume that the Prandtl number $\sigma = 1$ and the wall is thermally insulated ($\partial h/\partial\eta = 0$ when $\eta = 0$) . Then there exists an integral

$$h + {}^1/_2(u^2 + w^2) = {}^1/_2 \tag{4.7}$$

Finally, we shall assume that the dependence of the coefficient of viscosity on the enthalpy is linear ($\mu = 2h$) . Let us make a substitution of the variables in Equations (4.1); in place of the independent variables η and ζ introducing the variables s and ζ, where the A.A. Dorodnitsyn variable s is given by Equation

$$ds = R\,d\eta, \quad s = \int_0^\eta R\,d\eta, \quad \Phi(\zeta) = \int_0^\infty ds / R \tag{4.8}$$

It is essential that as we approach the outer surface of the boundary layer, i.e. when $\eta \to \Phi(\zeta)$, the quantity $s \to \infty$, whence follows the last equation of (4.8).

For dependent variables let us introduce two stream functions ψ and χ according to Equations

$$U = \partial\chi / \partial s, \qquad W - \zeta U = \partial\psi / \partial s \tag{4.9}$$

The equation of continuity can be written as

$$R\,(V - {}^3/_4\,\eta U) + (W - \zeta U)\,(\partial s\,/\,\partial\zeta)_\eta = -\,{}^5/_4\,\chi - \partial\psi\,/\,\partial s \qquad (4.10)$$

Taking into account Equations (4.6) to (4.10) the system of equations of the boundary layer (4.1) can be written in the form

$$-(\psi_\zeta + {}^5/_4\chi)\,\psi_{ss} + \psi_s\psi_{\zeta s} + \psi_s\chi_s + \varepsilon_1\,[1 - \chi_s^2(1+\zeta^2) - \psi_s^2 - 2\zeta\psi_s\chi_s] \times$$
$$\times\,[{}^1/_2\,\zeta + (1+\zeta^2)\,P'\,/\,P] = P\varepsilon_1^{-1}\psi_{sss}, \qquad \varepsilon_1 = \frac{\varkappa - 1}{2\varkappa}$$

$$-(\psi_\zeta + {}^5/_4\,\chi)\,\chi_{ss} + \psi_s\chi_{\zeta s} - \varepsilon_1\,[1 - \chi_s^2\,(1+\zeta^2) - \psi_s^2 - 2\zeta\psi_s\chi_s] \times$$
$$\times\,[{}^1/_2 + \zeta P'\,/\,P] = P\varepsilon_1^{-1}\chi_{sss} \qquad (4.11)$$

$$\Phi\Big(\frac{3}{4}\,\Phi - \zeta\Phi'\Big)^2 = \frac{2\varepsilon_1}{\varkappa+1}\int\limits_0^\infty [1 - \chi_s^2\,(1+\zeta^2) - \psi_s^2 - 2\psi_s\chi_s\zeta]\,dS$$

$$P = {}^1/_2\,(\varkappa+1)\,({}^3/_4\,\Phi - \zeta\Phi')^2$$

(where the subscripts s and ζ denote partial derivatives with respect to the corresponding quantities, whilst primes denote differentiation with respect to ζ).

The boundary conditions (4.5), after taking account of (4.9), assume the form

$$\psi\,(0,\zeta) = \chi\,(0,\zeta) = \psi_s\,(0,\zeta) = \chi_s\,(0,\zeta) = 0, \qquad \chi_s\,(\infty,\zeta) = 1$$
$$\psi_s\,(\infty,\zeta) = -\,\zeta \qquad (-\,\zeta \leqslant \zeta \leqslant \zeta_0) \qquad (4.12)$$

As a result of solution of the system of equations (4.11), with the boundary conditions (4.12), we determine the functions $\psi\,(s,\zeta)$, $\chi\,(s,\zeta)$, $P\,(\zeta)$, $\Phi\,(\zeta)$.

Let us seek the solution of Equations (4.11) in the form of the series

$$\psi = \sqrt{A_1}(\zeta_0-\zeta)^{1/4}\psi_1\,(\lambda) + (\zeta_0-\zeta)^{5/4}\,\psi_2\,(\lambda) + \ldots,\ \lambda = sA_1^{-1/2}(\zeta_0-\zeta)^{-1/4}$$
$$\chi = \sqrt{A_1}(\zeta_0-\zeta)^{1/4}\chi_1\,(\lambda) + (\zeta_0-\zeta)^{5/4}\,\chi_2\,(\lambda) + (\zeta_0-\zeta)^{9/4}\,\chi_3\,(\lambda) + \ldots$$
$$P = A_1\,(\zeta_0-\zeta)^{-1/2} + A_2\,(\zeta_0-\zeta)^{1/2} + A_3\,(\zeta_0-\zeta)^{3/2} + \ldots \qquad (4.13)$$
$$\Phi = B_1\,(\zeta_0-\zeta)^{3/4} + B_2\,(\zeta_0-\zeta)^{7/4} + B_3\,(\zeta_0-\zeta)^{11/4} + \ldots$$

Substituting these series in Equations (4.11) and (4.12), we obtain recurrent systems of ordinary differential equations with boundary conditions specified for $\lambda = 0$ and $\lambda = \infty$. The first system has the form

$$\psi_1\psi_1'' + 2\varepsilon_1\,(1+\zeta_0^2)\,L_1 = 4\varepsilon_1^{-1}\,\psi_1''', \qquad \psi_1\chi_1'' - 2\varepsilon_1\zeta_0 L_1 = 4\varepsilon_1^{-1}\chi_1'''$$
$$L_1 = 1 - (\chi_1')^2\,(1+\zeta_0^2) - (\psi_1')^2 - 2\zeta_0\psi_1'\chi_1' \qquad (4.14)$$
$$\psi_1\,(0) = \psi_1'\,(0) = \chi_1\,(0) = \chi_1'\,(0) = 0, \quad \psi_1'\,(\infty) = -\,\zeta_0, \quad \chi_1'\,(\infty) = 1$$

After solution of this system we determine the quantities

$$A_1 = \frac{3\varepsilon_1\zeta_0}{4}\,\frac{\sqrt{\varkappa+1}}{\sqrt{2}}\int\limits_0^\infty L_1\,(\lambda)\,d\lambda, \qquad B_1 = \frac{4\sqrt{2A_1}}{\sqrt{\varkappa+1}}\,\frac{1}{3\zeta_0} \qquad (4.15)$$

The subsequent systems will not be reproduced here; it is not difficult to see by substitution that if in Equations (4.13) we set

$$\psi_2 = \sqrt{A_1}\chi_1, \qquad \psi_i = 0 \quad (i \geqslant 3), \qquad \chi_i = A_i = B_i \equiv 0 \quad (i \geqslant 2) \tag{4.16}$$

then Expressions (4.13) satisfy Equation (4.11) and the boundary conditions (4.12).

Analysis of the expression for the pressure, taking account of (4.13) and (4.16), gives

$$p = \frac{P(\zeta)}{\sqrt{x}} = \frac{A_1}{\sqrt{x}\,(\cot\chi_0 - z/x)^{1/2}} = \frac{A_1\sqrt{\sin\chi_0}}{(x\cos\chi_0 - z\sin\chi_0)^{1/2}} = \frac{A_1\sqrt{\sin\chi_0}}{\sqrt{\xi}} \tag{4.17}$$

where ξ is the coordinate measured along the normal to the leading edge of the airfoil. Accordingly, the pressure (and hence also the other flow parameters) are just the same as if the flow was past a swept plate, the solution for the strong interaction on which is well known [9 and 10].

In view of the fact that Equations (4.11) are parabolic, having ζ as a characteristic, the solution, which is constructed from the leading edge, "does not know of" the presence of the other leading edge and does not satisfy the condition $W = 0$ in the plane of symmetry of the airfoil. (In the entropic layer (Section 2), on the contrary, by virtue of the equations being hyperbolic, the solution is constructed starting out from the condition $W = 0$ in the plane of symmetry). It can be shown that, for certain $\zeta = \zeta_*$ is nonviscous flow there arises a shock wave, hehind which the solution so constructed is not valid. The solution in the region $|\zeta| < |\zeta_*|$ will not be considered in the present paper.

In Fig. 4 and 5 are shown the results of a calculation of the functions $U(\frac{1}{2}\lambda\sqrt{\varepsilon_1})$ and $W(\frac{1}{2}\lambda\sqrt{\varepsilon_1})$ obtained by A.A. Bogacheva by means of the numerical Runge-Kutta solution of Equations (4.14) on an electronic computer. The values of χ_Δ and $\varkappa$ taken in the calculations are shown in the graphs.

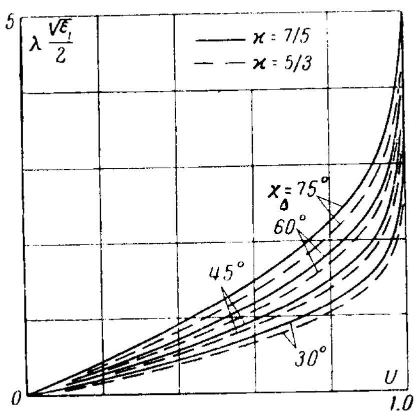

Fig. 4

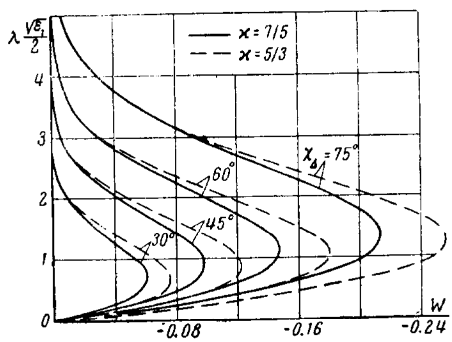

Fig. 5

From the calculations carried out it follows that the maximal value of $|W|$ when $\varkappa = 1.4$ and with χ_Δ, varying from 30° to 75°, lies in the range from 0.08 to 0.216, whilst if $\varkappa = 1.667$ it lies in the range 0.077 to 0.258. This confirms the estimates of Section 1 according to which $W \sim \varepsilon$. (It can be rigorously proved that when $\varepsilon \to 0$ the quantity $W \to 0$). As χ_Δ increases the quantity W increases, so that for highly swept airfoils the influence of the secondary flow can be important. It is vital to notice that in contradistinction to the case of the nonviscous entropy layer, in which the secondary flow is directed away from the plane of symmetry ($w > 0$ when $z > 0$), in the case of viscous flow past airfoils the secondary flow is directed towards the plane of symmetry ($w < 0$ when $z > 0$). From the projection of the momentum equation on the z-axis (the third equation of (4.1)) we find that, close to the plane of symmetry of the airfoil (when $z > 0$), the pressure must increase with decreasing z , in order to slow down the secondary flow and ensure fulfilment of the condition $w = 0$ when $z = 0$. Hence it can be concluded that the region $|\zeta| < |\zeta_*|$ close to the plane of symmetry of the airfoil, the solution for which is not considered, is a region of increased pressure, and not reduced pressure as in the case of the entropic layer.

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